

# Weak Hopf algebras corresponding to $U_q[sl_n]$

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## Abstract

We investigate the weak Hopf algebras of Li based on  $U_q[sl_n]$  and Sweedler's finite dimensional example. We give weak Hopf algebra isomorphisms between the weak generalisations of  $U_q[sl_n]$  which are “upgraded” automorphisms of  $U_q[sl_n]$  and hence give a classification of these structures as weak Hopf algebras. We also show how to decompose these examples into a direct sum which leads to unexpected isomorphisms between their algebraic structure.

# 1 Introduction

Since the introduction of quantum groups [1], the importance of Hopf algebras has been widely recognised in both mathematics and physics. Generalisations of Hopf algebras have been considered, usually motivated by some application in mathematical physics. The most well known example of the generalisations may be quasi-Hopf algebras where the coassociativity of a Hopf algebra is relaxed [2], but at the same time keeping the category of modules monoidal. A similar type of relaxation of coassociativity is also found in truncated quasi-Hopf algebras [3] and rational Hopf algebras [4]. This is where the notion of a weak co-product was introduced, such that  $\Delta(1) \neq 1 \otimes 1$ , and was motivated by the study of symmetries in low dimensional quantum field theory. One problem that arose was the fact that the dual of these structures was not associative, which led to further problems in defining crossed products and a double construction [5]. Although these issues have already been addressed in [6] and [7], the question still arose as to the possibility of defining a structure which could still provide non-integral dimensions for the quantum field theories in a similar way to the weak quasi-Hopf algebras [3], but at the same time be coassociative. This was the motivation behind defining the weak Hopf algebras of [5, 8, 9]. Since these are not bialgebras, but *almost* bialgebras [10], there were also axioms required to define a weak antipode, differing slightly from the usual ones of a Hopf algebra such that the category of finite dimensional modules was still monoidal, and also with a rigidity structure defined through a generalised antipode [9]. Another question then arose as to the possibility of defining a weak antipode on bialgebras. Li has introduced the notion of a weak Hopf algebra to mean a bialgebra on which is defined such a weak antipode [10, 11]. In this paper, we investigate these weak Hopf algebras as defined by Li.

The concept of a weak Hopf algebra is rather new. There are not many examples known, though several have appeared in the literature. One example is given by the semigroup algebra of any regular monoid which gives a generalisation of the well known group algebra [11]. The other known examples were given in [12] where the authors presented two weak Hopf generalisations of the quantised enveloping algebra  $U_q[sl_2]$ .

The purposes of this paper is two fold. First, we wish to propose some minor adjustments to the examples given in [12]. Second, we extend the construction to the case of other known Hopf algebras such as  $U_q[sl_n]$  [13] and Sweedler's Hopf algebra [14, 15]. It is also evident that we can define weak extensions of quantum superalgebras in a similar way. As a consequence, we shall have a lot of new nontrivial examples of weak Hopf algebras. We believe that for a deeper understanding of weak Hopf algebras as well having some insight into their applications, it is important to have various examples.

The paper is organised as follows. In section 2 we give a brief summary of the definition of weak Hopf algebra. Following that, section 3 has a closer look at the examples given in [12] and we propose slightly modified versions of these examples. We then realise in section 4 that for  $U_q[sl_n]$  there is a plethora of examples which leads us to finding isomorphisms between structures, thus giving a classification in some sense of weak Hopf algebras corresponding to  $U_q[sl_n]$ . The following section looks at weak extensions of Sweedler's famous finite dimensional Hopf algebra, where we also show that with our construction, in general we can decompose the weak Hopf algebra into a direct sum of the original bialgebra with some other subalgebra. This leads to “unexpected” algebra isomorphisms between structures in some cases.

## 2 Weak Hopf algebras

For the reader's convenience we recall the definition of a weak Hopf algebra in the sense of Li and Duplij [10, 12]. Let  $(H, \Delta, \varepsilon, m, u)$  be a bialgebra over a field  $K$ , where  $\Delta : H \rightarrow H \otimes H$  is the co-product,  $\varepsilon : H \rightarrow K$  is the co-unit,  $m : H \otimes H \rightarrow H$  the product and  $u : K \rightarrow H$  the unit of  $H$ . The following properties define  $H$ :

$$\begin{aligned} m(m \otimes \text{id}) &= m(\text{id} \otimes m), \\ m(u \otimes \text{id}) &= \text{id} = m(\text{id} \otimes u), \\ (\text{id} \otimes \Delta)\Delta &= (\Delta \otimes \text{id})\Delta, \\ (\varepsilon \otimes \text{id})\Delta &= \text{id} = (\text{id} \otimes \varepsilon)\Delta, \\ (m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\Delta \otimes \Delta) &= \Delta \circ m, \\ \varepsilon \otimes \varepsilon &= \varepsilon \circ m. \end{aligned}$$

Here  $\sigma : H \otimes H \rightarrow H \otimes H$  is the flip operator  $\sigma(h_1 \otimes h_2) = h_2 \otimes h_1$  for all  $h_1, h_2 \in H$ .

$H$  is a weak Hopf algebra if there is a weak antipode  $T : H \rightarrow H$  which is an algebra homomorphism satisfying the two conditions

$$T * \text{id} * T = T, \tag{2.1}$$

$$\text{id} * T * \text{id} = \text{id}, \tag{2.2}$$

with the convolution product  $*$  defined over maps on  $H$  by

$$a * b \equiv m(a \otimes b)\Delta : H \rightarrow H.$$

Note that the antipode of a Hopf algebra is a weak antipode due to the fact that  $u \circ \varepsilon : H \rightarrow H$  is the identity of the convolution product  $*$ . Recall that  $S : H \rightarrow H$  is an

antipode if it satisfies

$$S * \text{id} = u \circ \varepsilon, \quad (2.3)$$

$$\text{id} * S = u \circ \varepsilon. \quad (2.4)$$

For example, we see that

$$\begin{aligned} S * \text{id} &= u \circ \varepsilon \\ \Rightarrow \text{id} * S * \text{id} &= \text{id} * u \circ \varepsilon \\ \Rightarrow \text{id} * S * \text{id} &= \text{id}. \end{aligned}$$

In fact  $S$  needs only be a left or right antipode, meaning it satisfies only one of the two equalities (2.3) or (2.4), in order for it to be a weak antipode.

### 3 Weak $U_q[sl_2]$

In this section we give a summary of the examples of weak  $U_q[sl_2]$  presented in [12]. It is notable that the defining relations of the “J-weak” quantum algebra  $v\mathfrak{sl}_q(2)$  given in that paper can be simplified, and we give a minor adjustment to this example and show that it is in fact a weak Hopf algebra. We also give one other example generalising  $U_q[sl_2]$  which uses a mixture of the two examples from [12].

We remind the reader that the usual  $U_q[sl_2]$  relations to which we refer, in terms of the four generators  $E, F, K, K^{-1}$ , are as follows.

$$K^{-1}K = KK^{-1} = 1, \quad (3.1)$$

$$KEK^{-1} = q^2E, \quad (3.2)$$

$$KFK^{-1} = q^{-2}F, \quad (3.3)$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \quad (3.4)$$

The co-algebra structure (co-product  $\Delta$ , co-unit  $\varepsilon$ ) is given by

$$\begin{aligned} \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \\ \Delta(E) &= E \otimes K + 1 \otimes E, \\ \Delta(F) &= F \otimes 1 + K^{-1} \otimes F, \\ \varepsilon(E) &= \varepsilon(F) = 0, \\ \varepsilon(K^{\pm 1}) &= 1. \end{aligned}$$

It is clear that when we wish to determine the explicit action of the antipode, we apply the definition (2.3) and (2.4) to an arbitrary element in the algebra and solve. In

all cases we can solve explicitly due to the existence of the invertible group-like elements  $1, K, K^{-1}$ . The obvious first step in generalisation to the weak Hopf case would be to attempt to remove the invertibility of these elements. This was the main idea in [12] when generalising the above definition.

First, all weak extensions of  $U_q[sl_2]$  have generators  $E, F, K, \overline{K}$  satisfying

$$K\overline{K} = \overline{K}K \equiv J, \quad (3.5)$$

$$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K}, \quad (3.6)$$

$$EF - FE = \frac{K - \overline{K}}{q - q^{-1}}. \quad (3.7)$$

In what follows we usually write the generators with subscripts (following [12]) to differentiate the definitions.

**Definition 1** (from [12]):  $w\mathfrak{sl}_q(2)$  is the algebra generated by the four elements  $E_w, F_w, K_w, \overline{K}_w$  satisfying (3.5,3.6,3.7) along with the relations:

$$K_w E_w = q^2 E_w K_w, \quad (3.8)$$

$$\overline{K}_w E_w = q^{-2} E_w \overline{K}_w, \quad (3.9)$$

$$K_w F_w = q^{-2} F_w K_w, \quad (3.10)$$

$$\overline{K}_w F_w = q^2 F_w \overline{K}_w, \quad (3.11)$$

Here the invertibility of  $K$  and  $\overline{K}$  has been relaxed, and instead of the identity, the element  $J_w$  has been introduced. It can be seen that this element  $J_w$  satisfies

$$aJ_w = J_w a, \quad \forall a \in w\mathfrak{sl}_q(2).$$

To demonstrate this we check

$$\begin{aligned} E_w J_w &\stackrel{(3.5)}{=} E_w K_w \overline{K}_w \\ &\stackrel{(3.8)}{=} q^{-2} K_w E_w \overline{K}_w \\ &\stackrel{(3.9)}{=} K_w \overline{K}_w E_w \\ &\stackrel{(3.5)}{=} J_w E_w. \end{aligned} \quad (3.12)$$

A similar calculation is performed for  $F_w$  and the calculations for  $K_w$  and  $\overline{K}_w$  are trivial.

Also note that due to the relations (3.6),  $J_w$  is an idempotent. Namely,

$$J_w^2 = J_w.$$

The co-algebra structure is defined as follows. The co-product and co-unit are respectively given by

$$\begin{aligned}
\Delta_w(E_w) &= 1 \otimes E_w + E_w \otimes K_w, \\
\Delta_w(F_w) &= F_w \otimes 1 + \overline{K}_w \otimes F_w, \\
\Delta_w(K_w) &= K_w \otimes K_w, \\
\Delta_w(\overline{K}_w) &= \overline{K}_w \otimes \overline{K}_w, \\
\varepsilon_w(E_w) &= \varepsilon_w(F_w) = 0, \\
\varepsilon_w(K_w) &= \varepsilon_w(\overline{K}_w) = 1.
\end{aligned}$$

It can be verified that they are both algebra homomorphisms so that

$$\Delta_w(xy) = \Delta_w(x)\Delta_w(y)$$

and

$$\varepsilon_w(xy) = \varepsilon_w(x)\varepsilon_w(y)$$

for all  $x, y \in w\mathfrak{sl}_q(2)$ , thus preserving the defining relations. With this co-product a corresponding weak antipode can be determined by solving equations (2.1) and (2.2) with the above co-product. The only possible weak antipode in this case is

$$\begin{aligned}
T_w(1) &= 1, \\
T_w(K_w) &= \overline{K}_w, \\
T_w(\overline{K}_w) &= K_w, \\
T_w(E_w) &= -E_w \overline{K}_w, \\
T_w(F_w) &= -K_w F_w.
\end{aligned} \tag{3.13}$$

It can be shown that  $T_w$  is an algebra anti-homomorphism, that is,  $T(ab) = T(b)T(a)$ . Note that with the above bialgebra structure it is not possible to determine an antipode in the usual sense. As we mentioned previously, this was the motivation for relaxing (3.1) to (3.5) in order to provide weak antipodes which are not antipodes. For example, to solve the equation

$$S * \text{id}(K) = \varepsilon(K)1 \Rightarrow S(K)K = 1,$$

we would need an inverse of the element  $K$ .

Another possible definition given in [12] is the following.

**Definition 2** (from [12]):  $v\mathfrak{sl}_q(2)$  is the algebra generated by the four elements  $E_v, F_v, K_v, \overline{K}_v$  satisfying (3.5,3.6,3.7) along with the relations:

$$K_v E_v \overline{K}_v = q^2 E_v, \tag{3.14}$$

$$K_v F_v \overline{K}_v = q^{-2} F_v. \tag{3.15}$$

In this case,  $J_v = K_v \overline{K}_v$  satisfies the relation

$$J_v a = a J_v = a, \quad (3.16)$$

for  $a = E_v, F_v, K_v, \overline{K}_v$  (and hence  $J_v$ ). To demonstrate, we have

$$\begin{aligned} E_v J_v &\stackrel{(3.5)}{=} E_v K_v \overline{K}_v \\ &\stackrel{(3.14)}{=} q^{-2} K_v E_v \overline{K}_v K_v \overline{K}_v \\ &\stackrel{(3.6)}{=} q^{-2} K_v E_v \overline{K}_v \stackrel{(3.14)}{=} E_v \\ &\stackrel{(3.6)}{=} q^{-2} K_v \overline{K}_v K_v E_v \overline{K}_v \\ &\stackrel{(3.14)}{=} K_v \overline{K}_v E_v \\ &\stackrel{(3.5)}{=} J_v E_v. \end{aligned}$$

For the generator  $F_v$ , a similar calculation can be done. For the cases  $K_v$  and  $\overline{K}_v$ , the calculation is trivial. The most remarkable consequence of this property is that the analogue of defining relation (3.7) presented in [12] which was in the form

$$E_v J_v F_v - F_v J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}},$$

reduces to (3.7) by the above argument. Therefore in what follows we shall always use relation (3.7) and not the relation above.

The co-algebra structure for this second definition is as follows.

$$\begin{aligned} \Delta_v(E_v) &= J_v \otimes E_v + E_v \otimes K_v, \\ \Delta_v(F_v) &= F_v \otimes J_v + \overline{K}_v \otimes F_v, \end{aligned} \quad (3.17)$$

with the remaining actions coinciding precisely with the case of definition 1.

Moreover, relations (3.14) and (3.15) can be manipulated to those of definition 1. We demonstrate that

$$\begin{aligned} K_v E_v &\stackrel{(3.6)}{=} K_v \overline{K}_v K_v E_v \\ &\stackrel{(3.5), (3.16)}{=} K_v E_v \overline{K}_v K_v \\ &\stackrel{(3.14)}{=} q^2 E_v K_v. \end{aligned}$$

The other relations can be verified in a similar way. Although the co-product is different to that of definition 1, there exists a weak antipode which is the same as (3.13) in the case of definition 1.

This indicates that much of the discussion in [12] relating to  $vs\mathfrak{sl}_q(2)$  is redundant. However, we would like to make it clear that we consider the paper [12] rich in ideas and an inspiration to our current investigations.

There are other possibilities for defining weak extensions of  $U_q[sl_2]$ . These involve mixtures of definition 1 and definition 2 over the generators  $E, F$ . For example, we can say that one case is where  $E$  satisfies the relations (3.8), (3.9) and  $F$  satisfies (3.15), along with all the other relations common to both definition 1 and definition 2. The co-product would then have the action

$$\begin{aligned}\Delta(E) &= 1 \otimes E + E \otimes K, \\ \Delta(F) &= F \otimes J + \overline{K} \otimes F,\end{aligned}$$

along with the usual group-like co-product for  $K$  and  $\overline{K}$ . The weak antipode would still be the same as in definitions 1 and 2.

We can also swap this mixture of definitions and say that  $E$  satisfies those relations of definition 2, but  $F$  satisfies the relations of definition 1. This case is actually isomorphic to the first mixture, as we shall see later. In the section on Weak  $U_q[sl_n]$  we give a more formal way of notating such mixtures.

So we now have some clues as to how we may approach the problem of defining weak extensions of  $U_q[sl_n]$ . It is clear that there will be many possible combinations of generators satisfying either of the two definitions in the general case. This then begs the question: how would we know which mixtures of the two definitions lead to isomorphic algebras? To this end we have an important observation regarding some of the automorphisms of the original quantum algebra  $U_q[sl_n]$  which “lift up” to isomorphisms between weak Hopf structures. We shall look at these isomorphisms in more detail in the next section.

In general, we say that a generator satisfying the relations of definition 1 is of type 1, and is type 2 if it satisfies the relations of definition 2.

## 4 Weak $U_q[sl_n]$

### 4.1 Mixing definitions

For the case of  $ws\mathfrak{sl}_q(n)$ , which has simple generators  $E_i, F_i, K_i$  and  $\overline{K}_i$  ( $i = 1, \dots, n-1$ ), we can choose either definition 1 or 2 to describe the relations between any  $E_i$  and the  $K_j/\overline{K}_j$  and similarly for any  $F_i$ . This is what is meant by the word “mixture”. The relations satisfied by the generators are as follows, for all  $i, j$  unless specified otherwise;

$$K_i K_j = K_j K_i, \quad \overline{K}_i \overline{K}_j = \overline{K}_j \overline{K}_i, \quad K_i \overline{K}_j = \overline{K}_j K_i, \quad K_i \overline{K}_i = J,$$



$$\begin{aligned}
JK_j &= K_j J = K_j, \quad J\overline{K}_j = \overline{K}_j J = \overline{K}_j, \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - \overline{K}_i}{q - q^{-1}}, \\
E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 &= 0, \\
F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 &= 0, \\
E_i E_j &= E_j E_i, \quad F_i F_j = F_j F_i, \quad |i - j| \geq 2. \quad (4.1)
\end{aligned}$$

We also need to specify the relations between the  $E_i$  and the  $K_j$  for example. Let  $a_{ij}$  denote the Cartan matrix for  $sl(n)$ ,  $a_{ii} = 2$ ,  $a_{i,i\pm 1} = -1$  and zero otherwise. If  $E_i$  satisfies

$$K_j E_i = q^{a_{ij}} E_i K_j, \quad E_i \overline{K}_j = q^{a_{ij}} \overline{K}_j E_i, \quad \forall j, \quad (4.2)$$

we say  $E_i$  satisfies definition 1, or simply  $E_i$  is type 1. However, if  $E_i$  satisfies

$$K_j E_i \overline{K}_j = q^{a_{ij}} E_i, \quad \forall j, \quad (4.3)$$

we say  $E_i$  satisfies definition 2, or simply  $E_i$  is type 2. The same convention holds for  $F_i$  by replacing  $E_i$  with  $F_i$  and  $a_{ij}$  with  $-a_{ij}$  in the above relation. Notice also that  $J$  is defined for all  $i$ , so for example

$$J = K_i \overline{K}_i = K_j \overline{K}_j, \quad i \neq j.$$

The co-product has the following action;

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(\overline{K}_i) &= \overline{K}_i \otimes \overline{K}_i, \\
\Delta(E_i) &= \begin{cases} 1 \otimes E_i + E_i \otimes K_i, & E_i \text{ is type 1} \\ J \otimes E_i + E_i \otimes K_i, & E_i \text{ is type 2} \end{cases} \\
\Delta(F_i) &= \begin{cases} F_i \otimes 1 + \overline{K}_i \otimes F_i, & F_i \text{ is type 1} \\ F_i \otimes J + \overline{K}_i \otimes F_i, & F_i \text{ is type 2} \end{cases} \quad (4.4)
\end{aligned}$$

while the action of the co-unit is

$$\varepsilon(1) = \varepsilon(K_i) = \varepsilon(\overline{K}_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0.$$

The weak antipode  $T$  will always have the form

$$\begin{aligned}
T(1) &= 1, \\
T(K_i) &= \overline{K}_i, \\
T(\overline{K}_i) &= K_i, \\
T(E_i) &= -E_i \overline{K}_i, \\
T(F_i) &= -K_i F_i,
\end{aligned}$$

regardless of the type of the generators  $E_i$  and  $F_i$ .

In order to notate these mixtures for  $w\mathfrak{sl}_q(n)$  we use a binary notation, where a 1 indicates the use of a type 1 generator and a 0 indicates the use of a type 2 generator. We list the  $2(n-1)$  simple generators  $E_i$  and  $F_i$ , starting with the  $E_i$  followed by the  $F_i$ . We then write down a list of 0's and 1's in the order corresponding to the generators determined by their type. This then gives an integer from 0 to  $2^{2(n-1)} - 1$  in binary representation which contains all the information as to which particular mixture of definition we are using for the relations between the generators  $E_i$  and  $F_i$  and all the  $K_j/\overline{K_j}$ . We denote this integer  $d$  and the algebra is expressed as  $w\mathfrak{sl}_q^d(n)$ . In total there are  $2^{2(n-1)}$  possible mixtures for  $w\mathfrak{sl}_q^d(n)$ .

Note that we cannot have different definitions for the relations between the same generator with different  $K_i$ 's because the co-product could not possibly be consistent with those defining relations.

For example, in the case of  $w\mathfrak{sl}_q(4)$  we have the simple generators (not including the  $K_i$ ),  $E_1, E_2, E_3, F_1, F_2, F_3$ . Hence there are  $2^6 = 64$  different possibilities for relations with the  $K_i$ . The notation  $w\mathfrak{sl}_q^{43}(4)$  has the following meaning. Since the number 43 has the binary representation 101011, this is interpreted to mean that the simple generators  $E_1, E_3, F_2, F_3$  are type 1 with the remaining ones  $E_2, F_1$  being type 2. This information is determined by superimposing the list of binary digits  $\{1, 0, 1, 0, 1, 1\}$  with the list of simple generators in the order  $\{E_1, E_2, E_3, F_1, F_2, F_3\}$ .

It should also be noted that the algebra  $w\mathfrak{sl}_q^3(2)$  coincides with  $w\mathfrak{sl}_q(2)$  given in [12] (and in section 3 above) and the example  $v\mathfrak{sl}_q(2)$  of [12] is precisely  $w\mathfrak{sl}_q^0(2)$  in our notation.

## 4.2 Isomorphic structures

Now we look in more detail at the weak Hopf algebras of type  $U_q[sl_n]$  using mixtures of the two types of generators. In some cases where  $d_1 \neq d_2$ , there exists a weak Hopf algebra isomorphism  $w\mathfrak{sl}_q^{d_1}(n) \simeq w\mathfrak{sl}_q^{d_2}(n)$ . It is therefore worth investigating all possible isomorphisms in order to classify the weak extensions based on our criteria. As we shall see in this section, the isomorphisms are derived from a subset of the set of automorphisms on the algebra  $U_q[sl_n]$ . In other words, a subset of the automorphisms on  $U_q[sl_n]$  “lift up” to isomorphisms between the weak Hopf extensions. The reason only a subset can be considered, as we shall see later, is because some of the automorphisms of  $U_q[sl_n]$  lose their invertibility when upgraded to act on the weak  $w\mathfrak{sl}_q^d(n)$ , so therefore cannot be isomorphisms.

If  $(A, \Delta, \varepsilon, T)$  and  $(B, \Delta', \varepsilon', T')$  are weak Hopf algebras, then a weak Hopf algebra

isomorphism  $\psi : A \rightarrow B$  is an invertible algebra homomorphism satisfying

$$(\psi \otimes \psi) \circ \Delta = \Delta' \circ \psi, \quad (4.5)$$

$$\varepsilon = \varepsilon' \circ \psi, \quad (4.6)$$

$$\psi \circ T = T' \circ \psi. \quad (4.7)$$

For example, consider the algebra  $w\mathfrak{sl}_q^1(2)$ . This has generators  $E^{(1)}, F^{(1)}, K^{(1)}, \overline{K}^{(1)}$  (and  $1^{(1)}$ ) satisfying

$$\begin{aligned} K^{(1)}\overline{K}^{(1)} &= \overline{K}^{(1)}K^{(1)} \equiv J^{(1)}, \\ K^{(1)}\overline{K}^{(1)}K^{(1)} &= K^{(1)}, \quad \overline{K}^{(1)}K^{(1)}\overline{K}^{(1)} = \overline{K}^{(1)}, \\ K^{(1)}F^{(1)} &= q^{-2}F^{(1)}K^{(1)}, \quad \overline{K}^{(1)}F^{(1)} = q^2F^{(1)}\overline{K}^{(1)}, \\ K^{(1)}E^{(1)}\overline{K}^{(1)} &= q^2E^{(1)}, \\ E^{(1)}F^{(1)} - F^{(1)}E^{(1)} &= \frac{K^{(1)} - \overline{K}^{(1)}}{q - q^{-1}}, \end{aligned}$$

since the binary representation of  $1 = \{0, 1\}$  is superimposed with the list of generators  $\{E^{(1)}, F^{(1)}\}$  and so  $E^{(1)}$  is type 2 and  $F^{(1)}$  is type 1. Let us now consider the co-algebra structure of this algebra. The co-product  $\Delta$  and co-unit  $\varepsilon$  are respectively given by

$$\begin{aligned} \Delta(K^{(1)}) &= K^{(1)} \otimes K^{(1)}, \\ \Delta(\overline{K}^{(1)}) &= \overline{K}^{(1)} \otimes \overline{K}^{(1)}, \\ \Delta(E^{(1)}) &= J^{(1)} \otimes E^{(1)} + E^{(1)} \otimes K^{(1)}, \\ \Delta(F^{(1)}) &= F^{(1)} \otimes 1^{(1)} + \overline{K}^{(1)} \otimes F^{(1)}, \\ \varepsilon(E^{(1)}) &= \varepsilon(F^{(1)}) = 0, \\ \varepsilon(K^{(1)}) &= \varepsilon(\overline{K}^{(1)}) = 1. \end{aligned}$$

Now consider the map  $\psi : w\mathfrak{sl}_q^1(2) \rightarrow w\mathfrak{sl}_q^2(2)$  defined by the action

$$\begin{aligned} \psi(E^{(1)}) &= F^{(2)}, \\ \psi(F^{(1)}) &= E^{(2)}, \\ \psi(K^{(1)}) &= \overline{K}^{(2)}, \\ \psi(\overline{K}^{(1)}) &= K^{(2)}, \end{aligned}$$

where we have employed an obvious notation with superscripts. This map derives from the so-called Cartan involution on  $U_q[sl_2]$ . In the weak case it can be seen to be a weak Hopf algebra isomorphism since it preserves the generator type (that is, it is consistent with the defining relations) and is also consistent with equations (4.5)-(4.7).

In general the rule is that such an isomorphism must map a type 1 generator into a type 1 generator and similarly for type 2. We demonstrate the sort of calculation required to show consistency with the relations. Take for example

$$\begin{aligned}
\text{l.h.s.} = \psi(K^{(1)})\psi(E^{(1)})\psi(\overline{K}^{(1)}) &= \overline{K}^{(2)}F^{(2)}K^{(2)} \\
&= q^2\overline{K}^{(2)}(K^{(2)}F^{(2)}\overline{K}^{(2)})K^{(2)} \\
&= q^2F^{(2)} \\
&= q^2\psi(E^{(1)}) = \text{r.h.s.}
\end{aligned}$$

The other relations can be realised in a similar fashion. To demonstrate consistency with the co-product, we can use equation (4.5) to determine  $\Delta'$  in this case. For example, applying both sides of (4.5) to  $E^{(1)}$  gives

$$\begin{aligned}
(\psi \otimes \psi)\Delta(E^{(1)}) &= (\psi \otimes \psi)(J^{(1)} \otimes E^{(1)} + E^{(1)} \otimes K^{(1)}) \\
&= J^{(2)} \otimes F^{(2)} + F^{(2)} \otimes \overline{K}^{(2)}, \\
\Delta'(\psi(E^{(1)})) &= \Delta'(F^{(2)}).
\end{aligned}$$

This then gives the action of  $\Delta'$  on  $F^{(2)}$ . The remaining actions are

$$\begin{aligned}
\Delta'(E^{(2)}) &= E^{(2)} \otimes 1^{(2)} + K^{(2)} \otimes E^{(2)}, \\
\Delta'(K^{(2)}) &= K^{(2)} \otimes K^{(2)}, \\
\Delta'(\overline{K}^{(2)}) &= \overline{K}^{(2)} \otimes \overline{K}^{(2)}.
\end{aligned}$$

Note that this is not the co-product given in equations (4.4), but it is in fact the opposite co-product,  $\Delta^\sigma = \sigma \circ \Delta$  ( $\sigma$  being the flip operator), which we know from the theory of bialgebras is a perfectly acceptable one. It is also straightforward to verify (4.6) holds. Because the action of the weak antipode is dependent on the co-product,  $T'$  will be different to the one presented earlier. It is straightforward to verify that it does indeed exist, and that equation (4.7) is satisfied.

Although there are undoubtedly many other possibilities to extend  $U_q[sl_2]$  to a weak structure, the extensions presented in this paper based on that of [12] total three, namely,  $w\mathfrak{sl}_q^0(2)$ ,  $w\mathfrak{sl}_q^1(2) \simeq w\mathfrak{sl}_q^2(2)$  and  $w\mathfrak{sl}_q^3(2)$ . In each case the weak antipode has the same action, that of (3.13).

### 4.3 Number of unique structures

We now address the question of the number of possible weak Hopf algebra isomorphisms  $\psi : w\mathfrak{sl}_q^d(n) \rightarrow w\mathfrak{sl}_q^{d'}(n)$ ,  $d \neq d'$ .

It is well known that for  $U_q[sl_n]$  there are several types of automorphisms [16]. The most relevant to this paper are the Dynkin diagram automorphisms and the Cartan involution, since they give rise to Hopf algebra automorphisms and anti-automorphisms respectively. We consider maps  $\rho_d$  and  $\omega_d$  which have the same actions as the Dynkin diagram automorphism and the Cartan involution respectively, but applied to the weak Hopf algebra  $w\mathfrak{sl}_q^d(n)$ . These maps then become isomorphisms between weak Hopf structures. Their actions are given by

$$\begin{aligned}\rho_d(E_i^{(d)}) &= E_{n-i}^{(d')}, & \rho_d(F_i^{(d)}) &= F_{n-i}^{(d')}, & \rho_d(K_i^{(d)}) &= K_{n-i}^{(d')}, & \rho_d(\overline{K}_i^{(d)}) &= \overline{K}_{n-i}^{(d')}, \\ \omega_d(E_i^{(d)}) &= F_i^{(d'')}, & \omega_d(F_i^{(d)}) &= E_i^{(d'')}, & \omega_d(K_i^{(d)}) &= \overline{K}_i^{(d'')}, & \omega_d(\overline{K}_i^{(d)}) &= K_i^{(d'')},\end{aligned}$$

where the indices  $d$  (corresponding to the source),  $d'$  and  $d''$  (corresponding to the targets) are used to differentiate between the structures. Note that  $\rho_d$  and  $\omega_d$  map into different spaces which justifies the use of the different indices  $d'$  and  $d''$ . With these actions, it can be easily verified that

$$\rho_{d'} \circ \rho_d = \text{id}, \quad \omega_{d''} \circ \omega_d = \text{id}.$$

Lusztig [17] has also given a set of algebra automorphisms defined on the quantised enveloping algebras. However, when applied to our weak generalisations, they are found to be non-invertible and therefore are not isomorphisms between weak Hopf algebras.

Although we know of the existence of other algebra isomorphisms which exist between structures, the only known weak Hopf algebra isomorphisms are  $\rho_d$  and  $\omega_d$ . We will comment more on these algebra isomorphisms in section 5.

One important point is that  $\rho_d$  and  $\omega_d$  both preserve the generator type, so for example if  $E_i^{(d)}$  is a type 1 generator, so are  $E_{n-i}^{(d')}$  and  $F_i^{(d'')}$ . Therefore  $\rho_d$  and  $\omega_d$  must correspond to maps (say  $r_d$  and  $w_d$  respectively) defined on the non-negative integers such that

$$\begin{aligned}\rho_d : w\mathfrak{sl}_q^d(n) &\rightarrow w\mathfrak{sl}_q^{r_d(d)}(n), \\ \omega_d : w\mathfrak{sl}_q^d(n) &\rightarrow w\mathfrak{sl}_q^{w_d(d)}(n).\end{aligned}$$

Once we know the action of the maps  $r_d$  and  $w_d$ , this should allow us to be able to determine which structures are isomorphic and hence lead to a classification.

To this end, we write  $d$  in terms of its binary expansion

$$d = (d_0, d_1, \dots, d_{n-2} | d_{n-1}, \dots, d_{2n-3}),$$

where the bar separates the values representing the  $E_i$  and  $F_i$ , and where the  $d_i$  have values of either 0 or 1. Then  $r_d(d)$  and  $w_d(d)$  have the expansions

$$\begin{aligned}w_d(d) &= (d_{n-1}, \dots, d_{2n-3} | d_0, \dots, d_{n-2}), \\ r_d(d) &= (d_{n-2}, \dots, d_0 | d_{2n-3}, \dots, d_{n-1}).\end{aligned}$$

In terms of the components of the binary expansion we have

$$\begin{aligned} w_d(d_k) &= d'_{3n-4-k \bmod 2(n-1)}, \\ r_d(d_k) &= d''_{n-1+k \bmod 2(n-1)}. \end{aligned}$$

This simplifies the problem of determining isomorphic structures and allows us to explicitly count the number of unique structures for each  $n$ .

It is worth noting that  $\rho \circ \omega = \omega \circ \rho$ , so the only isomorphisms we need to consider are  $\rho$ ,  $\omega$  and  $\rho \circ \omega$  (we have dropped the subscripts for convenience). According to this prescription, there can only be at most four structures which are isomorphic. In some cases there could be two and in others there could be no isomorphisms. These cases are referred to below as degenerate. The situation can be summarised in the following diagram

$$\begin{array}{ccc} & r(d) \longrightarrow w \circ r(d) \\ & \uparrow \qquad \qquad \qquad \parallel \\ d & \swarrow \qquad \searrow \\ & w(d) \longrightarrow r \circ w(d) \end{array}$$

where in some cases the arrows could be equalities, in which case there would be one of the afore mentioned degeneracies. In order to count the number of unique (non-isomorphic) weak extensions of  $w\mathfrak{sl}_q^d(n)$ , we list all the possible degenerate cases;

- (1)  $w(d) = d$ ,
- (2)  $r(d) = d$ ,
- (3)  $r(d) = w(d)$ ,

and consider their intersection  $(1) \cap (2) \cap (3)$ , union  $(1) \cup (2) \cup (3)$  and their union's complement  $\overline{(1) \cup (2) \cup (3)}$  in order to count the total number of unique cases. We note that some of these cases will lead to exactly two isomorphic structures, and combinations of the above cases will lead to no isomorphic structures. We aim to separate each of these situations and then add the number of structures relating to each.

Case (1): The only  $d$  satisfying  $w(d) = d$  is of the form

$$d = (d_0, d_1, \dots, d_{n-2} | d_0, d_1, \dots, d_{n-2}).$$

Therefore the total number of cases satisfying case (1) is  $2^{n-1}$ .

Case (2): We separate this case into two cases corresponding to  $n$  being odd and even. For  $n = 2m + 1$  the only  $d$  satisfying case (2) is of the form

$$d = (d_0, d_1, \dots, d_{m-1}, d_{m-1}, \dots, d_0 | d_{2m}, d_{2m+1}, \dots, d_{3m-1}, d_{3m-1}, \dots, d_{2m}),$$

so there are  $2^{2m} = 2^{n-1}$  possibilities. For  $n = 2m$  the only  $d$  satisfying case (2) has the form

$$d = (d_0, \dots, d_{m-2}, d_{m-1}, d_{m-2}, \dots, d_0 | d_{2m-1}, \dots, d_{3m-3}, d_{3m-2}, d_{3m-3}, \dots, d_{2m-1}),$$

so there are  $2^{2(m-1)+2} = 2^{2m} = 2^n$  possibilities.

Case (3): The only  $d$  satisfying this case is of the form

$$d = (d_0, d_1, \dots, d_{n-2} | d_{n-2}, \dots, d_0),$$

so there are  $2^{n-1}$  possibilities.

(1)  $\cap$  (2)  $\cap$  (3): Once again we treat the case for  $n$  is even and odd separately. For  $n = 2m + 1$ ,  $d$  is of the form

$$d = (d_0, d_1, \dots, d_{m-1}, d_{m-1}, \dots, d_0 | d_0, d_1, \dots, d_{m-1}, d_{m-1}, \dots, d_0),$$

so there are  $2^m = 2^{(n-1)/2}$  possibilities. For  $n = 2m$ ,  $d$  is of the form

$$d = (d_0, \dots, d_{m-2}, d_{m-1}, d_{m-2}, \dots, d_0 | d_0, \dots, d_{m-2}, d_{m-1}, d_{m-2}, \dots, d_0),$$

giving  $2^m = 2^{n/2}$  possibilities.

(1)  $\cup$  (2)  $\cup$  (3): Combining the above three cases and subtracting twice their intersection gives the union, for which there are  $3 \cdot 2^{2m} - 2^{m+1}$  possibilities for  $n = 2m + 1$  and  $2^{2m+1} - 2^{m+1}$  possibilities for  $n = 2m$ . The compliment of the union then has  $2^{4m} - 3 \cdot 2^{2m} + 2^{m+1}$  possibilities for  $n = 2m + 1$  and  $2^{4m-2} - 2^{2m+1} + 2^{m+1}$  possibilities for  $n = 2m$ .

To calculate the exact number of unique structures we consider the fact that case (1) without cases (2) or (3) (and permutations) will have precisely 2 structures which are isomorphic, or put another way, isomorphic structures in these cases come in pairs. Therefore we need to half the number obtained above when counting the total number of structures. Similarly for the structures which do not fall into these degenerate cases. There will be exactly 4 isomorphic structures so we need to divide the number corresponding to (1)  $\cup$  (2)  $\cup$  (3) by 4.

Therefore the number of non-isomorphic structures, say  $Z_n$ , is

$$\begin{aligned} Z_{2m+1} &= \frac{2^{2m} - 2^m}{2} + \frac{2^{2m} - 2^m}{2} + \frac{2^{2m} - 2^m}{2} + 2^m + \frac{2^{4m} - 3 \cdot 2^{2m} + 2^{m+1}}{4} \\ &= 2^{4m-2} + \frac{3}{4} \cdot 2^{2m}, \\ Z_{2m} &= \frac{2^{2m-1} - 2^m}{2} + \frac{2^{2m} - 2^m}{2} + \frac{2^{2m-1} - 2^m}{2} + 2^m + \frac{2^{4m-2} - 2^{2m+1} + 2^{m+1}}{4} \\ &= 2^{4m-4} + 2^{2m-1}. \end{aligned}$$

Putting these two cases together gives

$$Z_n = 2^{n-4}(7 + (-1)^n + 2^n),$$

which is the number of unique weak Hopf structures corresponding to  $w\mathfrak{sl}_q^d(n)$ .

This formula for  $Z_n$  has been verified up to  $n = 10$  by directly applying the maps  $\rho$ ,  $\omega$  and  $\rho \circ \omega$  and then counting the number of unique structures. To give the reader an idea of the number of structures, we have the table below.

$n$	2	3	4	5	6	7	8	9	10
$Z_n$	3	7	24	76	288	1072	4224	16576	66048

We also list for  $n \leq 4$  all the values of  $d$ , putting isomorphic values in brackets  $\{, \}$ . For  $n = 2$  we have already determined that the values

$$d = 0, \{1, 2\}, 3$$

give the 3 unique structures. For  $n = 3$  the values of  $d$  for the 7 structures are

$$d = 0, \{1, 2, 4, 8\}, \{3, 12\}, \{5, 10\}, \{6, 9\}, \{7, 11, 13, 14\}, 15.$$

For  $n = 4$  the 24 values of  $d$  are

$$\begin{aligned} d = & 0, \{1, 4, 8, 32\}, \{2, 16\}, \{3, 6, 24, 48\}, \{5, 40\}, \{7, 56\}, \{9, 36\}, \{10, 17, 20, 34\}, \\ & \{11, 25, 38, 52\}, \{12, 33\}, \{13, 37, 41, 44\}, \{14, 28, 35, 49\}, \{15, 39, 57, 60\}, 18, \\ & \{19, 22, 26, 50\}, \{21, 42\}, \{23, 58\}, \{27, 54\}, \{29, 43, 46, 53\}, \{30, 51\}, \\ & \{31, 55, 59, 62\}, 45, \{47, 61\}, 63. \end{aligned}$$

All cases up to  $n = 10$  have been calculated, but are obviously too unwieldy to include in the article.

## 5 Direct sum decomposition and Sweedler's example

We now look in more detail at the algebraic structure and show that the upgraded quantised enveloping algebra automorphisms are not the only algebra isomorphisms between the various  $w\mathfrak{sl}_q^d(n)$ .

First recall Sweedler's example [14] of a finite dimensional Hopf algebra, denoted  $H$ .  $H$  is generated by elements  $I, G, X$  (where  $I$  is the identity element) satisfying the relations

$$\begin{aligned} G^2 &= I, \\ GX &= -XG, \\ X^2 &= 0. \end{aligned}$$



The co-product is given by

$$\begin{aligned}\Delta(G) &= G \otimes G, \\ \Delta(X) &= X \otimes G + I \otimes X,\end{aligned}$$

and the co-unit given by

$$\varepsilon(G) = 1 = \varepsilon(I), \quad \varepsilon(X) = 0.$$

The antipode  $S$  is given by the action

$$S(G) = G, \quad S(I) = I, \quad S(X) = GX.$$

Clearly  $H$  is 4 dimensional with basis  $\{I, G, GX, X\}$ .

In order to give an example of a weak Hopf algebra based on this structure with generators  $\{1, g, x\}$  (we now use lower case symbols), instead of using the relation  $g^2 = 1$ , we impose the relation  $g^3 = g$ . Moreover, we can choose either the relation  $gx = -xg$ , in which case we refer to  $x$  as a type 1 generator (analogous to the notion discussed at the end of section 3), or we can choose the relation  $gxg = -x$ , in which case we call  $x$  a type 2 generator. A type 2 generator is also a type 1 generator, but not conversely, since  $g^2 \neq 1$ .

For the first case, we choose  $x$  to be type 1. Denote the algebra by  $H_1$ . The following relations are satisfied.

$$\begin{aligned}g^3 &= g, \\ gx &= -xg, \\ x^2 &= 0,\end{aligned}$$

along with the same co-product and co-unit as in the usual Hopf case (given above). Solving equations (2.1) and (2.2) gives the weak antipode

$$\begin{aligned}T(1) &= 1, \\ T(g) &= g, \\ T(x) &= gx,\end{aligned}$$

which has the same action as the antipode from the Hopf case but nevertheless is not an antipode. The defining relations imply that  $H_1$  is 6 dimensional with basis

$$\{1, g, g^2, x, gx, g^2x\}.$$

Note that the element  $g^2$  is a central idempotent. This is easily verified with the defining relations.

It is with this example that we demonstrate explicitly how to obtain a direct sum decomposition for an algebra with a central idempotent. This procedure will then be extended to the case of  $w\mathfrak{sl}_q^d(n)$ .

$H_1$  has a direct sum decomposition

$$H_1 = H_1^0 \oplus H_1^1,$$

where  $H_1^0$  is the subalgebra with basis  $\{(1 - g^2)x, 1 - g^2\}$ , on which multiplication by  $g^2$  is zero (indicated in the superscript), and  $H_1^1$  is the subalgebra with basis  $\{g, g^2, gx, g^2x\}$  on which multiplication by  $g^2$  is the identity (also indicated in the superscript). In fact, these two subalgebras are determined by setting

$$\begin{aligned} H_1^0 &= (1 - g^2)H_1, \\ H_1^1 &= g^2H_1. \end{aligned} \tag{5.1}$$

It is straightforward to verify that the map  $\psi : H_1^1 \rightarrow H$  with the action

$$\begin{aligned} \psi(g) &= G, \\ \psi(g^2) &= I, \\ \psi(gx) &= GX, \\ \psi(g^2x) &= X, \end{aligned}$$

defines a weak Hopf algebra isomorphism, where  $I, G, X$  are the generators of the original Sweedler Hopf algebra  $H$ . Since  $H$  appears as a subalgebra of  $H_1$ , we can simply apply  $\psi^{-1} \otimes \psi^{-1}$  to the R-matrix of  $H$  (see [15]) to obtain an R-matrix  $\mathcal{R}$  of  $H_1$  satisfying

$$\begin{aligned} \mathcal{R}\Delta(a) &= \sigma \circ \Delta(a)\mathcal{R}, \quad \forall a \in H_1 \\ \mathcal{R}_{13}\mathcal{R}_{23} &= (\Delta \otimes \text{id})(\mathcal{R}), \\ \mathcal{R}_{13}\mathcal{R}_{12} &= (\text{id} \otimes \Delta)(\mathcal{R}). \end{aligned}$$

Such an R-matrix is then given by

$$\mathcal{R} = g^2 \otimes g^2 - 2p \otimes p + \alpha(g^2x \otimes g^2x - 2g^2x \otimes px + 2px \otimes px),$$

where  $p = (g^2 - g)/2$  and  $\alpha$  is an arbitrary parameter. This  $\mathcal{R}$  is not invertible, but it satisfies the regularity condition [12]

$$\mathcal{R}\hat{\mathcal{R}}\mathcal{R} = \mathcal{R}, \tag{5.2}$$

$$\hat{\mathcal{R}}\mathcal{R}\hat{\mathcal{R}} = \hat{\mathcal{R}}, \tag{5.3}$$

where

$$\hat{\mathcal{R}} = g^2 \otimes g^2 - 2p \otimes p + \alpha(g^2x \otimes g^2x - 2px \otimes g^2x + 2px \otimes px).$$

It should also be noted that the Sweedler Hopf algebra  $H$  also appears as a subalgebra of  $H_1$  with its generators defined by

$$\begin{aligned} I &= 1, \\ G &= 1 + g - g^2, \\ X &= (1 + \alpha g)gx, \end{aligned}$$

where  $\alpha$  is an arbitrary constant. However, this is just an observation and has no consequence to the results of our paper, since this subalgebra is only isomorphic to  $H$  as an algebra, not a bialgebra.

Now we look at the algebra  $H_2$ , which corresponds to the choice of the generator  $x$  to be of type 2. This implies the following relations;

$$\begin{aligned} g^3 &= g, \\ gxg &= -x, \\ x^2 &= 0. \end{aligned}$$

The only difference with the co-product in this case is with the action defined on the generator  $x$ , which is now given by

$$\Delta(x) = x \otimes g + g^2 \otimes x,$$

and the co-unit is the same as usual. The algebra  $H_2$  is 5 dimensional with basis  $\{1, g, x, gx, g^2\}$ . Note that  $g^2$  is a central idempotent. Defining

$$\begin{aligned} H_2^0 &= (1 - g^2)H_2, \\ H_2^1 &= g^2H_2, \end{aligned}$$

the decomposition

$$H_2 = H_2^0 \oplus H_2^1$$

still holds, where the superscripts still refer to the action of  $g^2$ , but now  $H_2^0$  has basis  $\{1 - g^2\}$  and  $H_2^1$  has basis  $\{g^2, g, x, gx\}$ .

Once again it is possible to verify that there exists a weak Hopf algebra isomorphism  $\varphi : H_2^1 \rightarrow H$  with the following action;

$$\begin{aligned} \varphi(g^2) &= I, \\ \varphi(g) &= G, \\ \varphi(x) &= X, \\ \varphi(gx) &= GX. \end{aligned}$$

In a similar way to both of the examples above, for a quantised enveloping algebra  $U \equiv U_q[sl_n]$ , its weak extension  $U_w$  and some other algebraic structure  $U_0$ , a decomposition of the form

$$U_w = U_0 \oplus U = (1 - J)U_w \oplus JU_w.$$

exists due to there being a central idempotent  $J$  whose existence derives from the relaxation of the invertibility of group-like elements in the algebra. In fact it is straightforward to prove the fact that for any  $d$ ,

$$U_q[sl_n] \simeq J.w\mathfrak{sl}_q^d(n).$$

This result is another way of stating Proposition 1 from the paper [12].

From another point of view, we could say that a weak extension is nothing but the original Hopf algebra plus some other algebra in which is contained all the information regarding the weak structure. In this case, it would help to know what conditions  $U_0$  would have to satisfy in order that  $U_w$  has a weak Hopf structure. This has not been the approach of this paper as we saw in the last section. Since there are no other weak Hopf algebra isomorphisms on the weak  $w\mathfrak{sl}_q^d(n)$ , the classification *as weak Hopf algebras* is complete. However, if we were to consider all possible algebra isomorphisms, this direct sum decomposition is important since it leads to discovering several “unexpected” isomorphisms which do not arise from the action of the automorphisms in the quantised enveloping algebra case.

Since we only need the presence of a central idempotent to achieve this direct sum decomposition, we can apply this idea to the weak extensions of  $U_q[sl_n]$  from the previous section, since the element  $J$  is always a central idempotent. However, it does not affect our classification of the weak Hopf algebra structure from the previous section. To demonstrate, we show that, rather unexpectedly, there is an algebra isomorphism  $\psi : w\mathfrak{sl}_q^{10}(3) \rightarrow w\mathfrak{sl}_q^9(3)$ .

We first apply the direct sum decomposition to  $U = w\mathfrak{sl}_q^{10}(3)$  and  $V = w\mathfrak{sl}_q^9(3)$  such that

$$U = (1 - J)U \oplus JU \equiv U_0 \oplus U_1$$

and similarly

$$V = (1 - J')V \oplus J'V \equiv V_0 \oplus V_1$$

Explicitly we have  $U_0$  generated by  $\langle (1 - J)E_1, (1 - J)F_1, 1 - J \rangle$  and  $U_1$  generated by  $\langle J, JE_1, JF_1, E_2, F_2, K_1, K_2, \overline{K}_1, \overline{K}_2 \rangle$ . For  $V$ , denoting its generators by a prime, we have  $V_0$  generated by  $\langle (1 - J')E'_1, (1 - J')F'_2, 1 - J' \rangle$  and  $V_1$  generated by  $\langle J', J'E'_1, E'_2, F'_1, J'F'_2, K'_1, K'_2, \overline{K}'_1, \overline{K}'_2 \rangle$ . It is straightforward to show that both  $U_1$  and

$V_1$  are isomorphic as (weak) Hopf algebras to  $U_q[sl(3)]$ . It is also easy to verify that  $U_0$  and  $V_0$  are both Abelian with the same number of generators and are therefore isomorphic as algebras. Combining these two facts leads to the isomorphism  $\psi : U \rightarrow V$ , the action of which is given by

$$\begin{aligned}
\psi(1) &= 1, \\
\psi(E_1) &= (1 - J')F'_2 + F'_1, \\
\psi(E_2) &= J'F'_2, \\
\psi(F_1) &= E'_1, \\
\psi(F_2) &= E'_2, \\
\psi(K_i) &= \overline{K}'_i, \\
\psi(\overline{K}_i) &= K'_i.
\end{aligned}$$

The map  $\psi$  is consistent with all the defining relations so is therefore an algebra homomorphism and it can be shown to have inverse

$$\begin{aligned}
\psi^{-1}(E'_1) &= F_1, \\
\psi^{-1}(E'_2) &= F_2, \\
\psi^{-1}(F'_1) &= JE_1, \\
\psi^{-1}(F'_2) &= (1 - J)E_1 + E_2, \\
\psi^{-1}(K'_i) &= \overline{K}_i, \\
\psi^{-1}(\overline{K}'_i) &= K_i.
\end{aligned}$$

Therefore  $\psi$  is an isomorphism.

If we set the action of the co-product  $\Delta$  for  $U$  and allow the freedom to choose the co-product  $\Delta'$  of  $V$  consistently with  $\psi$ , then we end up having to compare the action of  $(\psi \otimes \psi) \circ \Delta$  with  $\Delta' \circ \psi$ . Applying both of these maps to the generators will clearly give a non-coassociative  $\Delta'$ . Therefore the  $\psi$  only can be considered as an algebra isomorphism. However, it is uncertain whether or not this co-product  $\Delta'$  would define a quasi-bialgebra [2]. This certainly raises some interesting questions relating to whether or not these isomorphisms could correspond to some kind of Drinfeld twist. If so, then perhaps our classification is only a much smaller classification of the structures as quasi-bialgebras. This idea may warrant further investigation.

## 6 Concluding remarks

We have seen that it is possible to define weak extensions of  $U_q[sl_n]$  by only relaxing some of the relations in the original algebra. As we saw in the work of Li and Duplij [12], one nice way of doing this is to relax invertibility of the group-like elements to a more general regularity condition and also to impose one of two relations on the other generators. This allows us to define many examples.

One observation is that it is also possible to extend the definition of a quantised superalgebra [18] to the weak case by using the same idea of relaxing the invertibility of the generators  $K$  and  $\overline{K}$ . We demonstrate with the algebra  $wosp_q^d(2|1)$  which has generators  $\{K, \overline{K}, V_+, V_-\}$ . We define the parity of these generators to be  $p(K) = p(\overline{K}) = 0$ ,  $p(V_\pm) = 1$ .

The following relations are satisfied;

$$\begin{aligned} K\overline{K} &= \overline{K}K, & K\overline{K}K &= K, & \overline{K}K\overline{K} &= \overline{K}, \\ KV_\pm &= q^{\pm 1}V_\pm K, & \overline{K}V_\pm &= q^{\mp 1}V_\pm \overline{K}, \\ \{V_+, V_-\} &= -\frac{1}{4} \frac{K - \overline{K}}{q - q^{-1}}. \end{aligned} \tag{6.1}$$

Keeping in theme with the previous sections, if in addition the following relations are satisfied,

$$KX\overline{K} = q^{\pm 1}X$$

where  $X = V_+$  or  $V_-$ , then we call  $X$  a type 2 generator. Otherwise we call  $X$  a type 1 generator. This example is almost exactly like the case of  $w\mathfrak{sl}_q^d(2)$  in that we have the same notion of generators of type 1 and 2. The co-algebra structure is of the same form, and the only real difference is that the weak antipode is a graded algebra anti-homomorphism, so it satisfies  $T(ab) = (-1)^{p(a)p(b)}T(b)T(a)$ .

All the weak Hopf algebras given in this paper have non-cocommutative co-products. This implies existence of universal  $R$ -matrices that could give new solutions of quantum Yang-Baxter equations as mentioned in [10, 12]. One direction for future work is to investigate the form of such  $R$ -matrices. We expect the expressions would not be that different to those of the original Hopf algebra due to the direct sum decomposition of section 5. In fact, in section 5 we gave one possible  $R$ -matrix for the finite dimensional weak Hopf generalisation of Sweedler's well known example using these facts.

Section 5 also demonstrates the fact that there are many algebra isomorphisms between structures. In this article we did not investigate all possible isomorphisms, but instead gave a small existence proof that such isomorphisms do indeed exist. It would be interesting to classify these structures as algebras using this observation.

A question that arose during our investigation is one related to automorphisms and twisting, especially in the usual quantised enveloping algebra case. As we have already mentioned, the algebra automorphisms are well known for the quantised enveloping algebras, some of which are also bialgebra automorphisms. We are currently unaware of whether or not, corresponding to every algebra automorphism  $\psi : A \rightarrow A$ , there exists a twist element  $F \in A \otimes A$  such that

$$(\psi \otimes \psi)\Delta(a) = F.\Delta(\psi(a)).F^{-1}, \quad \forall a \in A.$$

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